Conformal Field Theories, Graphs and Quantum Algebras

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ABSTRACT
This article reviews some recent progress in our understanding of the structure of Rational Conformal Field Theories, based on ideas that originate for a large part in the work of A. Ocneanu. The consistency conditions that generalize modular invariance for a given RCFT in the presence of various types of boundary conditions –open, twisted– are encoded in a system of integer multiplicities that form matrix representations of fusion-like algebras. These multiplicities are also the combinatorial data that enable one to construct an abstract "quantum" algebra, whose $6j$- and $3j$-symbols contain essential information on the Operator Product Algebra of the RCFT and are part of a cell system, subject to pentagonal identities. It looks quite plausible that the classification of a wide class of RCFT amounts to a classification of "Weak C^*- Hopf algebras".

1 Introduction

For the last fifteen years or so, the study of Conformal Field Theory has been an amazingly active area of mathematical physics. Through all its connections with mathematics –infinite dimensional algebras, quantum deformations of algebras, integrable systems, combinatorial identities etc, and with its many applications in various fields of physics, from statistical mechanics to field and string theory, it offers a vivid evidence that exactly solvable physical problems may be an extraordinary source of enrichment and cross-fertilization, which is also what Barry McCoy has been so remarkably illustrating through his long and beautiful series of works. . .

The present article is devoted to a presentation in physical terms of some algebraic structures which are quite suitable for the description of rational conformal field theories (RCFT's) with or without boundaries, and which enable one to unravel their hidden symmetries and to derive new information. The main idea may be summarised as follows. Several consistency conditions encountered in RCFT lead to "nimreps", an acronym standing for *non-negative integer valued matrix representations*, of certain fusion al-

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gebras (and their extensions), and these nimreps may themselves be encoded in graphs. These graphs are in the simplest case Dynkin diagrams or their generalization; but more complicated patterns that we call Ocneanu graphs also appear. The graphs provide the combinatorial data needed for the construction of an algebra with two associative products, the Ocneanu Double Triangle Algebra (DTA), interpreted also as a pair of "weak C^*-Hopf algebras" in duality [5]. There is a growing evidence that the data needed to define and characterize a RCFT are made of a "cell system" attached to these graphs and subject to pentagonal identities. These cells determine the corresponding Ocneanu algebra, which can be interpreted as the quantum symmetry of the RCFT.

Most of the results of this review have been published in other papers [31, 32], in which the reader will find more details. The present work was inspired and deeply influenced by some recent work of A. Ocneanu [25, 26]. Ocneanu's work originates in the study of topological invariants of 3-manifolds and unravels the common features of their construction with problems encountered in RCFT. Recently the reverse path was followed in [15] to reformulate problems of RCFT in terms of the associated 3d Topological Field Theory. The precise connection between these approaches remains to be fully clarified.

2 Chiral RCFT and its data

Specification of a conformal theory begins with a certain number of chiral data. A *chiral algebra* $\mathfrak{A} \supseteq \mathrm{Vir}$ "containing" the Virasoro algebra is given. It may be Vir itself, or some current (untwisted affine) algebra, or some W-algebra, or more generally a Vertex Operator Algebra. Beside the Virasoro generators L_n, the additional generators of $\mathfrak{A}$ are denoted W_n. The assumption of *rationality* means that only a finite set $\mathcal{I}$ of irreducible representations (irreps) $\mathcal{V}_i$ has to be considered at a given value of the central charge. For each of these irreps, we introduce its *character* $\chi_i = \mathrm{tr}_{\mathcal{V}_i} q^{L_0 - c/24}$, c the Virasoro central charge. In a RCFT, these characters form a unitary representation of the modular group. If $q = e^{2\pi i\tau}$, they transform by a symmetric unitary matrix S under $\tau \to -1/\tau$: $\chi_i(\tau) = \sum_j S_{ij}\chi_j(-1/\tau)$, and by a diagonal unitary matrix under $\tau \to \tau + 1$.

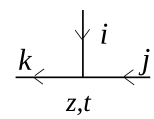
A fundamental notion is that of *fusion of the representations* $\mathcal{V}_i \star \mathcal{V}_j$ and of the fusion coefficients which arise from its decomposition into irreps $\mathcal{V}_i \star \mathcal{V}_j = \oplus_k \mathcal{N}_{ij}{}^k \otimes \mathcal{V}_k$. Here $\mathcal{N}_{ij}{}^k$ denotes a finite dimensional multiplicity space; by a small abuse of notations, we shall henceforth write this kind of decomposition as $\mathcal{V}_i \star \mathcal{V}_j = \oplus_k N_{ij}{}^k \mathcal{V}_k$, with $N_{ij}{}^k$ the integer dimension of the space $\mathcal{N}_{ij}{}^k$.

These integers give the structure constants of an associative commutative algebra which can be realized by the matrices $(N_i)_j{}^k$. It has an identity

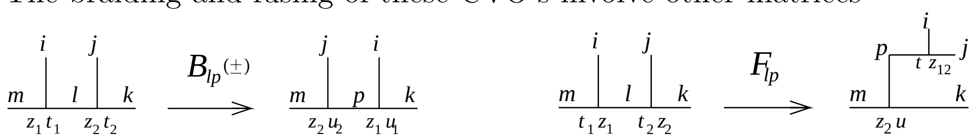
$N_1 = I$ corresponding to the "vacuum representation" $i = 1$, i.e. the one whose lowest eigenvalue of L_0 is 0. The structure constants are invariant under an involution map $i \to i^*$ s.t. ${N_{ji}}^1 = \delta_{ji^*}$ and are assumed to be given by the *Verlinde formula* ${N_{ij}}^k = \sum_{l\in\mathcal{I}} \frac{S_{il}S_{jl}S^*_{kl}}{S_{1l}}$ [38]. In the following, we shall make a repeated use of a rephrasing of this formula: for a fixed $l \in \mathcal{I}$, the ratios S_{il}/S_{1l} form a one-dimensional representation of the fusion algebra

$$\frac{S_{il}}{S_{1l}}\frac{S_{jl}}{S_{1l}} = \sum_{k\in\mathcal{I}} {N_{ij}}^k \frac{S_{kl}}{S_{1l}} \ . \tag{2.1}$$

These fusion coefficients are also interpreted as giving the dimensions of the spaces of chiral vertex operators (CVO's) $\phi^k_{ij;t}(z) \ : \ \mathcal{V}_j \mapsto \mathcal{V}_k \qquad t = 1,\cdots,{N_{ij}}^k$:



The braiding and fusing of these CVO's involve other matrices



Example: Take $\mathfrak{A} = \widehat{\mathfrak{g}}_k$, the current algebra based on the simple algebra $\mathfrak{g}$; $\mathcal{I}$ labels its set of integrable weights at level k; the S matrix is given in [21]; the F matrix turns out to be made of the $6j$- symbols of the $\mathcal{U}_q(\mathfrak{g})$ quantum algebra for $q = \exp 2i\pi/(h+k)$, with h the dual Coxeter number. In particular for $\widehat{sl}(2)_k$, $\mathcal{I} = \{1,2,\cdots,k+1\}$, and the k-dependent S-matrix is $S_{ij} = \sqrt{\frac{2}{k+2}}\sin\frac{\pi ij}{k+2}$.

3 RCFT on a plane or a torus

On a manifold without boundaries, like a plane or an infinite cylinder, or a torus, there are *two copies* of the chiral algebra $\mathfrak{A}$ at work, one relative to the holomorphic ("left") coordinates, the other to the antiholomorphic "right" ones. It is traditional to affect the right copy of the algebra and of its representations, characters etc with a bar. The Hilbert space of the theory is thus assumed to decompose as

$$\mathcal{H} = \oplus_{j,\bar{j}\in\mathcal{I}} \, Z_{j\bar{j}} \ \mathcal{V}_j \otimes \overline{\mathcal{V}}_{\bar{j}} \tag{3.1}$$

with a finite dimensional multiplicity space $Z_{j\bar{j}}$ for each pair $(j,\bar{j})$.

3.1 RCFT on a torus

On a torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, obtained by identifying the two ends of a finite segment of cylinder, and described by its modular ratio τ or nome

$q = e^{2i\pi\tau}$, a natural definition of the partition function of the system reads

$$Z = \mathrm{tr}_{\mathcal{H}} q^{L_0 - c/24} q^{*\bar{L}_0 - c/24} = \sum_{j\bar{j}} Z_{j\bar{j}} \, \chi_j(q) \, (\chi_{\bar{j}}(q))^* \tag{3.2}$$

As noticed by [6], this partition function must be modular invariant. Moreover, the unicity of vacuum imposes that $Z_{11} = 1$.

We are thus led to a well posed (but difficult!) problem ...

Problem: *Classify the Modular Invariants* $Z_{j\bar{j}} \in \mathbb{Z}_{\geq 0}$, $\quad Z_{11} = 1$.

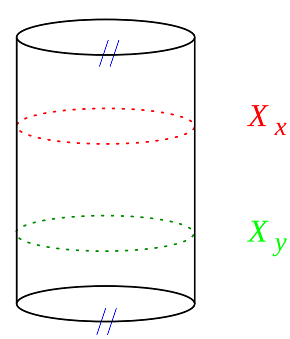
As the representation of the modular group is unitary, $Z_{j\bar{j}} = \delta_{j\bar{j}}$ is always a solution, the "diagonal modular invariant", in which all representations of $\mathcal{I}$ appear once in a left-right symmetric way: the corresponding "diagonal theory" has thus only spinless primary fields. The problem is to find all the other solutions. Unfortunately in spite of a constant flow of new results due to the tenacity of T. Gannon [19], this approach has its limitations. Modular invariance is just a necessary condition, but is not sufficient to fully specify the theory, and other constraints may rule out some candidate solutions. It is thus appropriate to look for additional restrictions.

For later reference, we recall that modular invariants come in two types. Type I are block-diagonal invariants, of the form $Z = \sum_a |\sum_i b_a^i \chi_i|^2$, and are interpreted as the diagonal invariants for some larger chiral algebra $\mathfrak{A}^{\mathrm{ext}} \supset \mathfrak{A}$ whose characters decompose with branching coefficients $b_a^i \in \mathbb{Z}_{\geq 0}$. Type II invariants are obtained from some invariant of type I, its "parent", through a twist of the right components with respect to the left by some automorphism ζ of the fusion rules of $\mathfrak{A}^{\mathrm{ext}}$: $Z = \sum_a \sum_{i\bar{i}} b_a^i b_{\zeta(a)}^{\bar{i}} \chi_i \chi_{\bar{i}}^*$. (See the third reference in [3] for examples beyond this simple classification.)

3.2 Torus again, but with defect lines

Let us now imagine that on the cylinder we introduce non-local operators attached to non-contractible loops. These operators X may be called defect or twist lines, in the context of lattice models they have received the name of "seams", since they modify the nature of the boundary conditions as we close the cylinder into a torus [31].

We demand that these operators, which act in the Hilbert space $\mathcal{H}$ of equation (3.1), commute with the action of $\mathfrak{A} \otimes \bar{\mathfrak{A}}$



$$\begin{aligned} [L_n, X] &= [\bar{L}_n, X] = 0 \\ [W_n, X] &= [\overline{W}_n, X] = 0 \ . \end{aligned} \tag{3.3}$$

The interpretation of the former condition is that the action of X is invariant under infinitesimal deformation of the contour, and the latter looks like a natural extension to all the generators $W_n, \overline{W}_n$ of the full chiral algebra.

Now, Schur's lemma leads to a complete characterization of these X. Their most general form is a linear combination of $P^{(j,\bar{j};\alpha,\alpha')}$ which act as projectors between equivalent copies of $\mathcal{V}_j \otimes \bar{\mathcal{V}}_{\bar{j}}$

$$(\mathcal{V}_j \otimes \bar{\mathcal{V}}_{\bar{j}})^{\alpha'} \mapsto (\mathcal{V}_j \otimes \bar{\mathcal{V}}_{\bar{j}})^{\alpha}, \qquad \alpha, \alpha' = 1, \cdots, Z_{j\bar{j}} \ .$$

The number of independent X is thus $\sum Z_{j\bar{j}}^2 = \mathrm{tr}\, Z Z^T$. We write a basis of such X in the form

$$X_x = \sum_{j\bar{j},\alpha,\alpha'} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')}}{\sqrt{S_{1j}S_{1\bar{j}}}}\, P^{(j,\bar{j};\alpha,\alpha')} \,, \tag{3.4}$$

where Ψ is an invertible square matrix of dimension $\sum Z_{j\bar{j}}^2$ and the denominator $\sqrt{SS}$ is introduced for later convenience. Among these operators stands as a trivial solution the identity operator, which we label by $x = 1$ and for which $\Psi_1^{(j,\bar{j};\alpha,\alpha')} = \delta_{\alpha\alpha'} \sqrt{S_{1j}S_{1\bar{j}}}$.

We then write the partition function on the torus with the insertion of two defect lines in two alternative ways. On the one hand

$$Z_{x|y} := \mathrm{tr}_{\,\mathcal{H}}\, X_x^\dagger X_y \ \tilde{q}^{L_0 - c/24} \tilde{q}^{*\,\bar{L}_0 - c/24} \,, \tag{3.5}$$

where $\tilde{q} = \exp \frac{-2\pi i}{\tau}$, is expressed through (3.4) as

$$Z_{x|y} = \sum_{\substack{j,\bar{j} \in \mathcal{I} \\ \alpha,\alpha'=1,\cdots,Z_{j\bar{j}}}} \frac{\Psi_x^{(j,\bar{j};\alpha,\alpha')\,*}\,\Psi_y^{(j,\bar{j};\alpha,\alpha')}}{S_{1j}S_{1\bar{j}}}\ \chi_j(\tilde{q})\,(\chi_{\bar{j}}(\tilde{q}))^* \ . \tag{3.6}$$

On the other hand, the system is also described by a Hilbert space

$$\mathcal{H}_{x|y} = \oplus_{i,\bar{i} \in \mathcal{I}}\, \widetilde{V}_{i\bar{i}^*;\,x}{}^{y}\, \mathcal{V}_i \otimes \overline{\mathcal{V}}_{\bar{i}} \ ,$$

with some multiplicities $\widetilde{V}_{ij;x}{}^{y} \in \mathbb{Z}_{\geq 0}$, and in particular if the defects are trivial,

$$\widetilde{V}_{i\bar{i}^*;\,1}{}^{1} = Z_{i\bar{i}} \ , \tag{3.7}$$

the modular invariant matrix. Then the same partition function reads

$$Z_{x|y} = \mathrm{tr}_{\,\mathcal{H}_{x|y}}\, q^{L_0 - c/24}\, q^{*\,\bar{L}_0 - c/24} = \sum_{i,\bar{i} \in \mathcal{I}} \widetilde{V}_{i\bar{i};\,x}{}^{y}\ \chi_i(q)\,(\chi_{\bar{i}^*}(q))^* \ . \tag{3.8}$$

Note that the summation in (3.6) runs over the set $\{(j,\bar{j};\alpha,\alpha'),\, \alpha,\alpha' = 1,\cdots,Z_{j\bar{j}}\}$ describing for $\alpha = \alpha'$ the physical spectrum (3.1) with its multiplicities, while in (3.8) it runs over all pairs of irreps of $\mathcal{I}$. Comparing the two expressions of $Z_{x|y}$ and assuming the sesquilinear combinations $\chi_i\chi_{\bar{i}}^*$ to be independent (which is justified only after a slight generalization of this argument involving non-specialized characters [2, 31]), we find the consistency condition

$$\widetilde{V}_{i\bar{i};x}{}^{y} = \sum_{j,\bar{j},\alpha,\alpha'} \frac{S_{ij}\,S_{\bar{i}\bar{j}}}{S_{1j}S_{1\bar{j}}}\ \Psi_x^{(j,\bar{j};\alpha,\alpha')}\ \Psi_y^{(j,\bar{j};\alpha,\alpha')\,*}\,, \qquad i,\bar{i}\in\mathcal{I}\,. \tag{3.9}$$

For $x = 1 = y$ this relation boils down to the modular invariance property $Z = S\,Z\,S^*$ of the torus partition function. To proceed, we make the additional assumption that the matrix Ψ in the change of basis of solutions of (3.4) $P^{(j,\bar{j};\alpha,\alpha')} \to X_x$ is unitary. Then (3.9) is the diagonalization formula of $\widetilde{V}_{i\bar{i}}$, whose eigenvalue $S_{ij}S_{\bar{i}\bar{j}}/S_{1j}S_{1\bar{j}}$ has multiplicity $Z_{j\bar{j}}^2$. We now recall (2.1) and conclude that the $\widetilde{V}$ matrices form a "nimrep" of the double fusion algebra

$$\widetilde{V}_{i_1j_1}\widetilde{V}_{i_2j_2} = \sum_{i_3,j_3} N_{i_1i_2}{}^{i_3} N_{j_1j_2}{}^{j_3}\ \widetilde{V}_{i_3j_3}\ . \tag{3.10}$$

The study of RCFT in the presence of defect lines has thus led us to a new

Problem: *Classify the nimreps $\widetilde{V}$ of the Double Fusion Algebra*
(with the constraints that their spectrum is dictated by the modular invariant matrix $Z_{i\bar{i}}$ as in (3.9) and that $\widetilde{V}_{i\bar{i}^*;1}{}^{1} = Z_{i\bar{i}}$ and $\widetilde{V}_{ij}^T = \widetilde{V}_{i^*j^*}$.)

Remark. From a physical perspective, these defect lines or seams generalize cases which were known before and which were constructed using symmetries of an underlying lattice model. For example, a closed non-contractible disorder line of the Ising model, which flips all the couplings along the edges that it crosses, is an actual realization of a certain operator X_x with x corresponding to a "simple current". Using the $\mathbb{Z}_2$ invariance of the model, this line may be freely moved around along one of the two cycles of the torus, in a discrete analogue of the condition (3.3). This observation may offer a partial justification to the denomination "quantum symmetries" that A. Ocneanu has given to objects labelled by x [25]. Hopefully, the work in progress [9, 10] on the actual realization of all the operators X_x in lattice RSOS models will improve our intuition about these operators.

4 RCFT on half-plane or annulus

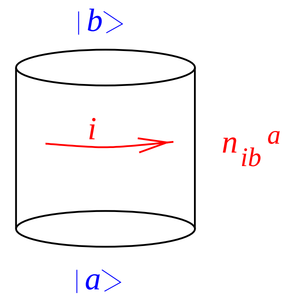
A consistent CFT must lead to a sensible quantum theory on any two-dimensional manifold (world-sheet), irrespective of whether it has or not boundaries. In the presence of boundaries, however, we expect constraints of a new nature to emerge, as we are probing different features of the theory.

4.1 Cylinder: boundary states, Cardy condition

Let's review the case of a cylinder with boundary conditions a and b (yet to be determined) imposed at its two ends. As the discussion of this case is quite parallel to the one of section 3.2, we shall be brief, see [1, 2]. Once again, we shall impose the consistency between two pictures:

(1) On the one hand, the system may be described in terms of boundary states $|a\rangle$, $|b\rangle$ in $\mathcal{H}$ satisfying

$$(L_n - \bar{L}_{-n})|a\rangle = 0$$



(and likewise for $|b\rangle$) and their generalizations to other generators W_n. One proves (again by a suitable application of Schur's lemma) that there exists a canonical ("Ishibashi") such state $|j,\alpha\rangle\rangle$ in each $\mathcal{V}_j \otimes \mathcal{V}_{\bar{j}}$ iff $j = \bar{j}$ (up to automorphisms, see [2] and references therein) and $Z_{jj} \neq 0$. We use $\alpha = 1, \cdots, Z_{jj}$ to label a basis of such states. The most general boundary state is thus written as a linear superposition

$$|a\rangle = \sum_{j,\alpha=1,\cdots,Z_{jj}} \frac{\psi_a^{(j,\alpha)}}{\sqrt{S_{1j}}} |j,\alpha\rangle\rangle \ .$$

(2) On the other hand, the Hilbert space of the theory in the upper half-plane supports the action of a single copy of Vir or $\mathfrak{A}$ and thus decomposes as

$$\mathcal{H}_{ba} = \oplus_{i\in\mathcal{I}}\, n_{ib}{}^a\ \mathcal{V}_i \tag{4.1}$$

with integer multiplicities $n_{ib}{}^a$. We now compute the partition function of the system on a finite cylinder in the two pictures: evolution between the boundary states $|a\rangle$ and $\langle b|$ on the one hand, or periodic evolution in the Hilbert space $\mathcal{H}_{ab}$ on the other. Consistency gives the **Cardy equation** [7]:

$$n_{ia}{}^b = \sum_{j,\alpha} \psi_a^{(j,\alpha)} \psi_b^{(j,\alpha)\,*} \frac{S_{ij}}{S_{1j}}. \tag{4.2}$$

We again assume ψ to be a unitary matrix (reflecting the completeness of the set of boundary states) and invoke (2.1) to conclude that

$$n_i n_j = \sum_k N_{ij}{}^k n_k\,, \qquad (n_i)^T = n_{i^*}\,. \tag{4.3}$$

Hence the $\{n_i\}$ form a nimrep of the fusion algebra * and this leads us to

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*The Verlinde algebra nimreps (4.3), introduced in [12], were conjectured there to provide the coefficients of cylinder partition functions. The completeness requirement was first stressed in [33] in a particular example. However a key step of the attempted derivation of (4.3) there (the identification of the boundary field multiplicity $n_{ia}{}^b$ with that of bulk field couplings) is incorrect.

yet another ...

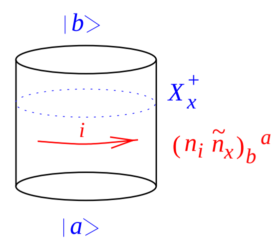
Problem: *Classify the nimreps n of the fusion algebra*
(with the constraint that their spectrum is dictated by the (diagonal part of the) modular invariant Z_{jj} as in (4.2), and subject to the symmetry $n_{i^*} = n_i^T$.)

This problem has already been investigated for a while. Since in our setting the spectrum is restricted to that dictated by the modular invariant we automatically discard as unphysical any would be solution of the system (4.3) not consistent with modular invariance: see below in sect. 5.2 an example of such a spurious solution. On the other hand there may be more than one solution corresponding to the same modular invariant. Some of those may have to be discarded by further requirements, see the discussions below.

Before we show explicit solutions of these nimreps, let us complete our guided tour by combining defects and boundaries.

4.2 Cylinder and Defect Line

It is very natural to combine the two situations above and to look at the spectrum of the theory when both defect lines and boundaries are present. One finds that on a cylinder with one defect line $X_x^\dagger$ and boundary states a and b, a new set of integer multiplicities occurs in the decomposition of $\mathcal{H}_{bx|a} = \oplus_{i\in\mathcal{I}} (n_i\, \tilde{n}_x)_b{}^a \mathcal{V}_i$



$$\tilde{n}_{ax}{}^b = \sum_{j,\alpha,\beta} \psi_a^{(j,\alpha)} \frac{\Psi_x^{(j,j;\alpha,\beta)}}{\sqrt{S_{1j} S_{1\bar{j}}}} \psi_b^{(j,\beta)*} \,. \tag{4.4}$$

The structure of equation (4.4) looks familiar but note that in contrast with the cases encountered so far, it doesn't express the full diagonalization of the matrices $\tilde{n}$, at least whenever some $Z_{jj} > 1$, resulting in a non-diagonal sum over α, β. In these conditions, the $\tilde{n}$ **do not** commute a priori. Indeed the $\tilde{n}$ form a nimrep of a new, associative but non-commutative, algebra, endowed with a $*$ involution

$$\begin{aligned}
\tilde{n}_x \tilde{n}_y &= \sum_z \widetilde{N}_{xy}{}^z \tilde{n}_z \\
\widetilde{N}_{yx}{}^z &= \sum_{j,\bar{j};\alpha,\beta,\gamma} \Psi_y^{(j,\bar{j};\alpha,\beta)} \frac{\Psi_x^{(j,\bar{j};\beta,\gamma)}}{\sqrt{S_{1j} S_{1\bar{j}}}} \Psi_z^{(j,\bar{j};\alpha,\gamma)*} \\
\widetilde{N}_x \widetilde{N}_y &= \sum_z \widetilde{N}_{xy}{}^z \widetilde{N}_z \,.
\end{aligned} \tag{4.5}$$

Since $\tilde{n}_{ax}{}^b$ are interpreted as multiplicities both (4.4) and (4.5) have to give non-negative integers. Note that it is quite non-trivial that bases Ψ

and ψ may be found such that the integrality of the $\tilde{n}$, $\widetilde{N}$ holds true. As a consequence of (3.9,4.5) and of the unitarity of Ψ, the $\widetilde{V}_{i\bar{i};x}{}^{y}$ taken for fixed $i, \bar{i}$ satisfy

$$\widetilde{V}_{x}{}^{y} = \sum_{z} \widetilde{N}_{xz}{}^{y}\, \widetilde{V}_{1}{}^{z} \ . \tag{4.6}$$

Physically, the algebra with the structure constants $\widetilde{N}_{yx}{}^{z}$ in (4.5) describes the *fusion of defect lines*; (4.6) says that once the $\widetilde{N}$ are known, insertion of an arbitrary number of defect lines reduces to a single one; and the fact that this algebra is non-commutative expresses that defect lines cannot be freely swapped along the cylinder. This non-commutativity takes place whenever the Hilbert space (3.1) contains multiplicities larger than 1. The simplest statistical mechanical model in which this is expected and where it has now been observed on the lattice is the 3-state Potts model [10].

5 From Nimreps to Graphs and Cells

5.1 The system of nimreps

Let us summarize where we are. We have found that the spectrum of a RCFT in various "environments" is described by a set of multiplicity matrices which have the remarkable property of forming nimreps of fusion algebras

$$\begin{aligned} N_i N_j &= \sum_k N_{ij}{}^{k} N_k \\ n_i n_j &= \sum_k N_{ij}{}^{k} n_k \\ \widetilde{V}_{i_1 j_1} \widetilde{V}_{i_2 j_2} &= \sum_{i_3, j_3} N_{i_1 i_2}{}^{i_3} N_{j_1 j_2}{}^{j_3}\ \widetilde{V}_{i_3 j_3} \\ \tilde{n}_x \tilde{n}_y &= \sum_z \widetilde{N}_{xy}{}^{z} \tilde{n}_z \\ \widetilde{N}_x \widetilde{N}_y &= \sum_z \widetilde{N}_{xy}{}^{z} \widetilde{N}_z \,. \end{aligned} \tag{5.1}$$

Assuming in addition that the matrices n_i and $\tilde{n}_x$ commute, the second and the fourth of these equations can be combined into

$$(n_i \tilde{n}_x)\,(n_j \tilde{n}_y) = \sum_{k,z} N_{ij}{}^{k}\, \widetilde{N}_{xy}{}^{z}\ (n_k \tilde{n}_z) \ . \tag{5.2}$$

These various sets of matrices cannot be studied completely independently as they are connected by various types of relations: multiplicities of eigenvalues of n_i, $\widetilde{V}_{ij}$ given in terms of Z, commutation relations $[n_i, \tilde{n}_x] = 0$, etc. The system (5.1) simplifies in the *diagonal* cases in which $Z_{ij} = \delta_{ij}$. Then the ranges of all labels coincide and (5.1) is solved with $n = \tilde{n} = \widetilde{N} = N$, $\widetilde{V}_{ij} = N_i N_j$.

5.2 Graphs

The (integer valued) matrices n_i , $\widetilde{V}_{ij}$ are conveniently encoded as graphs. Each of them may be regarded as the adjacency matrix of a graph whose vertices are labelled by the set of indices of the matrix at hand. As these matrices form a nim*rep* of an algebra and are thus reconstructed from the generators of that algebra, it is sufficient to draw the graph(s) corresponding to the generator(s). For example, for $\widehat{sl}(2)$, it is sufficient to give N_2, n_2, and $(\widetilde{V}_{21}, \widetilde{V}_{12})$ to reconstruct all N_i, n_i and $\widetilde{V}_{ij}$.

Graphs of n. Consider first the simple case of $\widehat{sl}(2)_k$. By (4.2), n_2 must have eigenvalues $S_{2j}/S_{1j} = 2\cos\pi j/(k+2) < 2$, for some $j \in \mathcal{I}$ hence be the adjacency matrix graph of an *A*-*D*-*E* Dynkin diagram or a "tadpole" $A_{2n}/\mathbb{Z}_2$. Tadpoles are ruled out by the spectral condition as their eigenvalues do not match the *ADE* classification of modular invariants. For the case of $\widehat{sl}(3)$, enough nimreps have been found along the years [23, 12, 2] to match Gannon's list of modular invariants [19] and it is now believed [26] that the list is complete. See [2] for that list and [16] for another systematic study in the case of simple current modular invariants.

Given a nimrep $\{n_i\}$ of matrices of size $p \times p$ and its set of graphs, it proves useful to introduce their *associative graph algebra*: it is realized by $p \times p$ matrices $\hat{N}_a = \{\hat{N}_{ba}{}^c\}$, $\hat{N}_a \hat{N}_b = \sum_c \hat{N}_{ab}{}^c \hat{N}_c$, such that $\hat{N}_1 = I_p$, the identity matrix, for some special vertex denoted 1. The $\hat{N}_a$ are requested to satisfy

$$n_i \hat{N}_a = \sum_b n_{ia}{}^b \hat{N}_b \ . \tag{5.3}$$

Thus, if matrix $\hat{N}_a$ is attached to vertex a, multiplication by n_i gives the sum of its "neighbours" on the graph of n_i. For all known type I theories, (see end of sect. 3.1), one observes the following properties: (i) such $\hat{N}$ matrices may be found with non negative integer entries, see [3, 32] for a (block) spectral decomposition of these –in general non-commutative– matrices; (ii) a subalgebra of this graph algebra associated with a subset T of the vertices gives the fusion algebra of the extended chiral algebra $\mathfrak{A}^{\mathrm{ext}}$ [13]. Thus the subset T of vertices is in one-to-one correspondence with the set of representations of $\mathfrak{A}^{\mathrm{ext}}$. One may further require that (iii) the integers $n_{i1}{}^a$, $a \in T$ describe the multiplicity of the representation i in the representation a of $\mathfrak{A}^{\mathrm{ext}}$ and thus determine the corresponding block-diagonal modular invariant, i.e., $n_{i1}{}^a = b^i_a$ in the notations of sect. 3.1. This is found in many type I theories [13, 30] but this rules out some exceptional solutions of (4.3), which are otherwise consistent with the spectral properties: an example is provided by the nimrep of $\widehat{sl}(3)_9$ called $\mathcal{E}_3^{(12)}$ in the list of [2]. The properties (i-iii) of type I theories are incorporated in the subfactor approach [39, 3, 4], where (4.3), (5.3) are solved recursively for n_i and $\hat{N}_a$, starting from the $a=b=1$ element of the matrix relation (4.3), and using the data $N\,,N^{\rm ext}\,,{n_{i1}}^a\,,\ a\in T$ of $\mathfrak{A}\,,\mathfrak{A}^{\rm ext}$. The algorithm extends to type II with an appropriate choice of ${n_{i1}}^1$, see also section 3.5 of [2]. Note that (5.3) implies

$$ {n_{ia}}^b = \sum_c {\hat{N}_{ac}}{}^b\, {n_{i1}}^c\,, \tag{5.4} $$

in clear analogy with (4.6). In particular for any a, ${n_{ia}}^a\geq {n_{i1}}^1$ in any type I theory. This inequality was recently proposed in [18] (see also the discussion in [20]), as a constraint for selecting "physical" solutions among the nimreps $\{n_i\}$ and is non trivial from the point of view adopted in that paper. The previous discussion shows that in the present framework it is not very helpful, as for type I theories it follows from the positivity of the graph algebra, assumed here, while for type II it fails, as may be explicitly checked on examples. Inserting (5.4) into (4.1) provides an alternative decomposition of the half plane Hilbert space (and of the corresponding partition function) [2]

$$ \mathcal{H}_{ab} = \oplus_c\ {\hat{N}_{ac}}{}^b\ \hat{\mathcal{V}}_c\,, \tag{5.5} $$

in terms of the "twisted" (or "solitonic") representations

$$ \hat{\mathcal{V}}_c = \oplus_{i\in\mathcal{I}}\ {n_{i1}}^c\ \mathcal{V}_i\,, \tag{5.6} $$

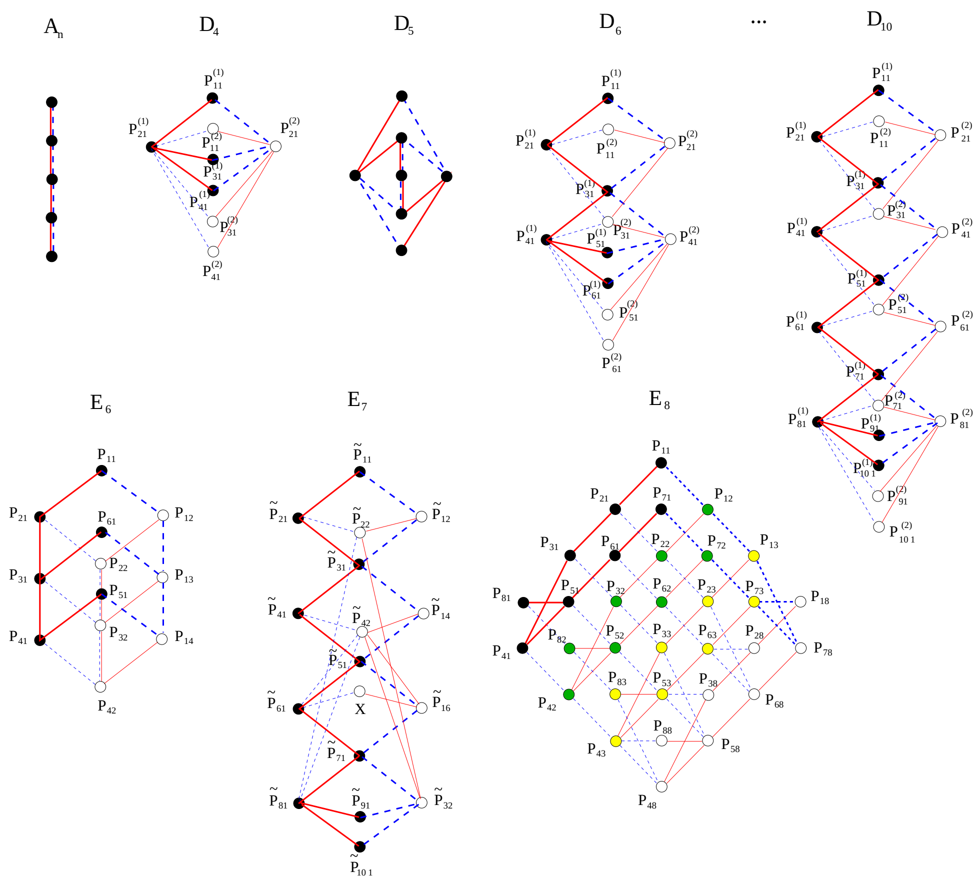
which are true representations of $\mathfrak{A}^{\rm ext}$ for $c\in T$ in type I. The integers in (5.6) (or in (5.4)), the intertwining matrix ${n_{i1}}^c$, can be found for all ADE $sl(2)$ graphs in Table I of [12]. The $\hat{N}_c$ fusion algebra of the type I $sl(2)$ graphs $A\,,D_{\rm even}\,,E_6\,,E_8$ is at the heart of a notion of "finite subgroups of the quantum group $U_q(sl(2)$" for q root of unity in the recent paper [22], where the decompositions (5.6) appear with a different interpretation. On the other hand the approach of Ocneanu to the same problem is based on the fusion algebra $\widetilde{N}_x$ of the $\widetilde{V}$ graphs, to which we now turn.

Graphs of $\widetilde{V}$. The graphs of $\widetilde{V}$ may be less familiar, they were introduced by Ocneanu first in the $\widehat{sl}(2)$ case [25], see [26, 3, 4, 32, 11] for further developments or discussions. They are depicted on the Table below for $\widehat{sl}(2)$: the two graphs of $\widetilde{V}_{21}$ and of $\widetilde{V}_{12}$ are drawn on the same chart, with edges of the former in solid (red) lines, those of the latter in broken (blue) lines. The matrices $\widetilde{V}_{i1}$ and $\widetilde{V}_{1i}$ satisfy (4.3), i.e. form a nimrep of the same type as n_i, but their spectrum is in general different. In particular, the graph of $\widetilde{V}_{21}$, say, is in general made of several connected components of ADE type. This results from (3.9) which tells us that in this $\widehat{sl}(2)$ case $\widetilde{V}_{21}$ has eigenvalues

$2\cos j\pi/h$ with multiplicity $\sum_{\bar{j}} Z^2_{j\bar{j}}$. It follows from the explicit form of the Z's that

$$\begin{array}{rl} A_n & \widetilde{V}_{21} \sim A_n \\ D_{2\ell+1} & \widetilde{V}_{21} \sim A_{4\ell-1} \\ D_{2\ell} & \widetilde{V}_{21} \sim D_{2\ell} \oplus D_{2\ell} \\ E_6 & \widetilde{V}_{21} \sim E_6 \oplus E_6 \\ E_7 & \widetilde{V}_{21} \sim D_{10} \oplus E_7 \\ E_8 & \widetilde{V}_{21} \sim E_8 \oplus E_8 \oplus E_8 \oplus E_8 \end{array} \tag{5.7}$$

where $\sim$ means the equality in a certain basis. The same holds for $\widetilde{V}_{12}$, but in a different basis, and it is more difficult to see how these two graphs intertwine into the patterns of the Table. It seems reasonable to conjecture that in general the graphs of $\widetilde{V}_{i1}$ and of $\widetilde{V}_{1i}$ always contain a connected component isomorphic to the graph of n_i relative to the "parent" theory.



Note that $\widetilde{V}_{ij}\widetilde{N}_x = \sum_z \widetilde{V}_{ij\,x}{}^z \widetilde{N}_z$ in analogy with (5.3): the $\widetilde{N}$ matrices form the graph algebra of the Ocneanu graph, and it is thus natural to attach $\widetilde{N}_x$ to the vertex x of the latter. Moreover, one proves that there exist two special vertices denoted 2 and $\bar{2}$ such that

$$\widetilde{N}_2 = \widetilde{V}_{21} \qquad \widetilde{N}_{\bar{2}} = \widetilde{V}_{12}\,. \tag{5.8}$$

This follows from the observation that (3.10, 3.7) gives $\sum_x (\widetilde{V}_{21;1}{}^x)^2 = Z_{11} + Z_{13} = 1$, as $Z_{13} = 0$ for all $\widehat{sl}(2)$ modular invariants. Thus there exists a unique vertex $x := 2$ such that $\widetilde{V}_{21;1}{}^x = 1$. Then from (4.6), $\widetilde{V}_{21} = \widetilde{N}_2$ and a similar property for $\widetilde{V}_{12}$. This discussion extends for all $\widehat{sl}(N)$ to $\widetilde{V}_{f,1}$ where f is the fundamental representation with a single box Young tableau.

In contrast with their counterparts $\hat{N}_a$, the matrices $\widetilde{N}_x$ are nonnegative integer valued for the type I and type II theories alike. According to the previous conjecture on the existence of a connected component of Ocneanu graph isomorphic to the graph of the n_i, in type I cases, the $\hat{N}$ algebra is represented isomorphically by a subalgebra of the Ocneanu graph algebra $\widetilde{N}$, while in type II cases it is the parent $\hat{N}$ graph algebra which is a subalgebra of the parent $\widetilde{N}$ algebra; recall that $\widetilde{N}$ is required to be positive in general. Thus for the Ocneanu graphs the type I versus type II distinction is less conspicuous, and the existence in all cases of a fusion ring associated with each of these graphs plays an important role in the construction of the corresponding quantum algebra, see below.

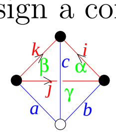
Instead of attaching matrix $\widetilde{N}_x$ to vertex x in the graph of $\widetilde{V}_{21}$ and $\widetilde{V}_{12}$, one may also attach to it the matrix $\widetilde{V}_1{}^x$. Upon multiplication by N_2, the usual fusion matrix, acting on the first label i of $\widetilde{V}_{ij}$, one gets, by virtue of (3.10), $\sum_{i'} N_{2i}{}^{i'} \widetilde{V}_{i'j;1}{}^x = \sum_y \widetilde{V}_{21;y}{}^x \widetilde{V}_{ij;1}{}^y$. Likewise, action of N_2 on the second label is represented by $\widetilde{V}_{12}$. The annotations on the figures of the Table encode some information on this representation: the matrices $V_{ij;1}{}^x$, forming the "toric structure" of Ocneanu [26], may be expressed in terms of quadratic combinations of the n matrices, $\widetilde{V}_{ij} = \sum n_i \times n_j$, where we are deliberately vague about the summation, the correspondence between the indices of $\widetilde{V}$ and those of the n's etc. See section 7 and Appendix B of [32] for a detailed discussion. More precisely, one must distinguish between the type I theories for which this holds true, and the type II ones, here the cases $D_{2\ell+1}\,, E_7$, for which this is verified in terms of the n matrices of the parent type I theory ($A_{4\ell-1}$ and D_{10} respectively). The above factorization of the matrices $\widetilde{V}_1{}^x$ is paralleled by a factorisation of the multiplicities $\tilde{n}_x$ and $\widetilde{N}$, which are expressed in terms of the $n_i\,, \hat{N}$ data of either the graph G or its parent graph.

5.3 Cells

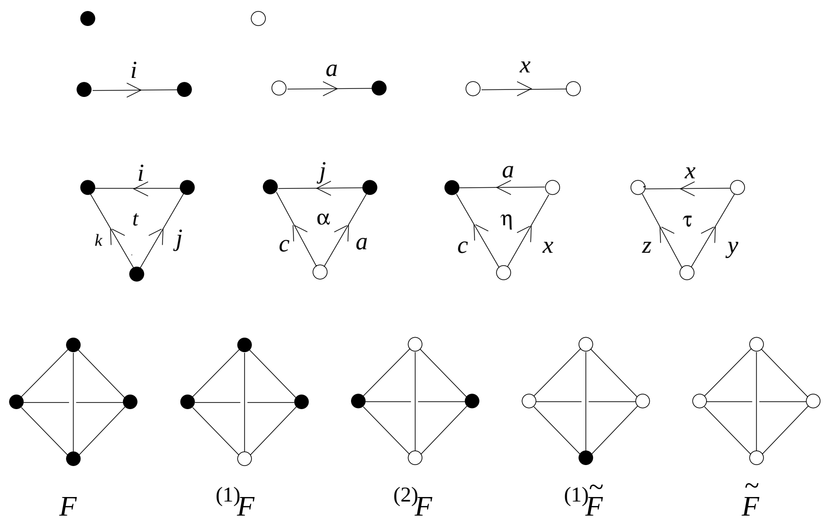
The previous multiplicities (or graphs) specify the spectrum of the RCFT. But as is familiar, beyond the spectrum, we need additional information on the "Operator Product Algebra" to determine fully the theory and be able to compute all correlation functions. This requires to attach other quantities, which following Ocneanu we call *cells*, to these graphs.

Together, as we shall see in the next section, these data enable one to construct a new quantum algebra $\mathcal{A}$ (and its dual $\hat{\mathcal{A}}$). Its description is conveniently achieved by first constructing a simplicial 3-complex made with the elements of the next figure [24, 5].

The 1-simplices (edges) carry labels of the nature discussed in the beginning of these notes: i is a representation label, a a boundary state, x a defect. Each 2-simplex (triangle) comes with a multiplicity label which takes $N_{ij}{}^k$, $n_{ja}{}^c$, $\tilde n_{ax}{}^c$, $\widetilde N_{xy}{}^z$ values, respectively. Finally to each of the 3-simplices (which we have not decorated with their indices, for more clarity), one must assign a complex number called a cell (the value of a 3-cochain).

For example 

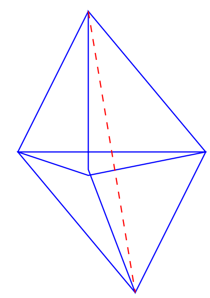
$= {}^{(1)}F_{bk}\left[\begin{smallmatrix} i & j\\ c & a\end{smallmatrix}\right]^{\beta\, t}_{\alpha\,\gamma}$, $t = 1,\cdots, N_{ij}{}^k$, $\alpha = 1,\cdots, n_{ib}{}^c$, etc.



In the same way as in section 2 where F was expressing a change of basis in the fusion of CVO's, all the cells should be regarded as invertible matrices. The fact that F , ${}^{(1)}F$, ${}^{(1)}\widetilde F$, $\tilde F$ are square matrices is precisely one of the conditions of the big system (5.1). For example ${}^{(1)}F_{bk}\left[\begin{smallmatrix} i & j\\ c & a\end{smallmatrix}\right]^{\beta\, t}_{\alpha\,\gamma}$ has (b,α,γ) as a "row index" and (k,β,t) as a "column index" and

$$\sum_{b,\alpha,\gamma} 1 = \sum_b n_{ja}{}^b n_{ib}{}^c = \sum_k N_{ij}{}^k n_{ka}{}^c = \sum_{k,\beta,t} 1 \ .$$

Similarly the spectral decompositions of the mutually commuting n_i and $\tilde n_x$ imply the relation $\sum_x \tilde n_{ax}{}^{a'} \tilde n_{b' x^*}{}^b = \sum_i n_{ia}{}^b n_{i^* b'}{}^{a'}$ which ensures the invertibility of ${}^{(2)}F$.



This system of cells may be chosen to satisfy unitarity constraints. More crucially, it must satisfy the "Big Pentagon equation" (a name adopted in [5]), namely a set of 6 quintic identities of the form

$$\begin{aligned} FFF &= FF\,, \qquad F\,{}^{(1)}F\,{}^{(1)}F = {}^{(1)}F\,{}^{(1)}F\,, \\ {}^{(1)}F\,{}^{(2)}F\,{}^{(2)}F &= {}^{(1)}F\,{}^{(2)}F, \quad \text{etc}\cdots \end{aligned}$$

and their dual counterparts. It may be pictorially interpreted as expressing independence of the cochain with respect to the splitting of a double tetrahedron into two or three tetrahedra: see figure. There, all assignments of the two colours to the vertices are allowed, but the number of identities is reduced by the unitarity of the cells.

Some of these cells and their pentagon equations have a direct physical interpretation, notably the second pentagon above expresses the associativity of the boundary fields product, with the cells ${}^{(1)}F$ serving as OPE coefficients [35, 2]. See [32] for their lattice model interpretation. There is an increasing evidence that the consistency of the RCFT requires the existence of the whole system, which leads us to a formulation of the ...

Refined problem : *Find a system of nimreps (5.1) leading to cells consistent with the pentagon identities.*

In fact very little is known about this problem. In the diagonal theories all the cells coincide, in an appropriate gauge, with the fusing matrices F, explicitly known in the $\widehat{sl}(2)$ cases. Runkel has computed the ${}^{(1)}F$ cells in the $\widehat{sl}(2)$ D cases [36] and some of the matrix elements of ${}^{(1)}F$ in the exceptional cases have been worked out earlier in the context of their lattice interpretation. In the simpler $\widehat{sl}(2)$ $D_{\rm odd}$ case one has $\widetilde{N} = N\,, \tilde{n} = n$, and accordingly ${}^{(1)}\widetilde{F} = {}^{(1)}F\,, \widetilde{F} = F$ while ${}^{(2)}F$ is expressed bilinearly in terms of ${}^{(1)}F$.

Since in the type I cases the $\widetilde{N}$ algebra contains a subalgebra isomorphic to the graph algebra $\hat{N}$ one can restrict in a first step all labels in the two pentagon identities for the dual $3j$- and $6j$-symbols ${}^{(1)}\widetilde{F}$ and $\widetilde{F}$ to a subset identified with the boundary set $\{x = a\}$. Then these two relations coincide if we identify these particular matrix elements of the two sets of cells ${}^{(1)}\widetilde{F}$ and $\widetilde{F}$. In this way one introduces $6j$-symbols $\hat{F}_{bd}\begin{bmatrix} a & e \\ c & f \end{bmatrix}$ with all six labels given by boundary indices; the triangles of these tetrahedra are determined by the graph fusion algebra multiplicities $\hat{N}_{ab}{}^{c}$. †

We recall from sect. 5.2 that a subset T of the boundary labels can be identified with the representations of the extended fusion algebra. Accor-

†These $\hat{F}$, "hidden" in the set of dual $3j$- and $6j$-symbols of the Ocneanu algebra, are to be compared with the $6j$-symbols of the "boundary category" in [18].

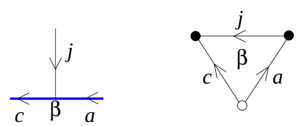
dingly $\hat{F}$, with all 6 labels further restricted to T, can be identified with the $6j$-symbols (the fusing matrices) $F^{\rm ext}$ of chiral vertex operators of the extended theory.

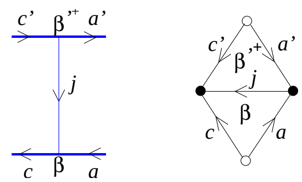
The matrices $\hat{N}_a$ enter as building blocks in the expressions for the type I multiplicities $\tilde{n}$ and $\widetilde{N}$ [32], and it is natural to expect that this property can be "lifted" to the dual cells, ${}^{(1)}\widetilde{F}$, $\widetilde{F}$, expressing them in terms of some $\hat{F}$. One can further speculate that the $\hat{F}$ themselves may be found in terms of the $3j$-symbols, extending to the cells the relation (5.4) and the algorithm in [39] for solving (4.3), (5.3).

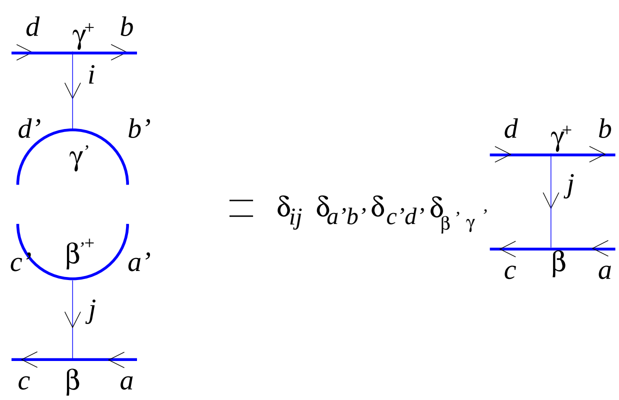
6 The Double Triangle Algebra and (B)CFT

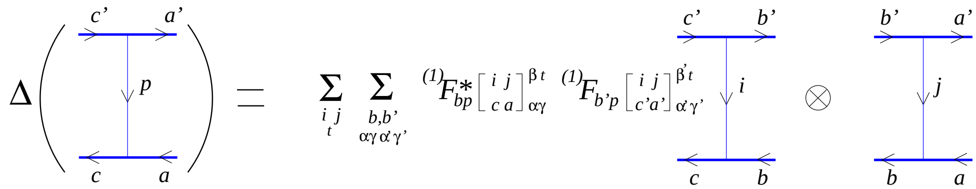
In this last section we sketch how knowledge of the system (5.1) and of the cells enables one to construct a finite dimensional quantum algebra with peculiar properties.

As a first step we construct vector spaces V^j of dimension $m_j = \sum_{a,c} n_{ja}{}^c$ with an orthogonal basis $|e^{j,\beta}_{ca}\rangle$, $\beta = 1, \cdots, n_{ja}{}^c$. They correspond to the second type of triangles above, depicted dually by generalized vertices: see figure. The precise normalisation of these basis vectors which involves quantum dimensions and components of the Perron-Frobenius vector $\psi^{(1)}$ is of no importance for the present qualitative discussion. Then one considers the matrix algebra $\mathcal{A} = \oplus_{j\in\mathcal{I}} End\, V^j \cong \oplus_{j\in\mathcal{I}} {\rm Mat}_{m_j}$, where each block is generated by the "double triangles" $e^{(ca)\,,(c'a')}_{j;\beta,\beta'} =$ const. $|e^{j,\beta}_{ca}\rangle\langle e^{j\,,\beta'}_{c'a'}|$, i.e., states in $V^j \otimes V^{j^*}$. The unit for the matrix product is given by $1 = \sum_{j,a,c,\beta} e^{(ca)\,,(ca)}_{j;\beta,\beta}$. This algebra admits a coassociative coproduct defined in terms of the $3j$-symbols ${}^{(1)}F$, a $*$- operation, counit and antipode; the product and the coproduct can be depicted as on the picture below.









These operations satisfy a slightly weakened version of the axioms of Hopf algebras, defining (finite dimensional) *weak* C^**- Hopf algebras* [5]: in particular the coproduct does not preserve the identity, $\Delta(1) \neq 1 \otimes 1$, and two of the axioms on the counit and the antipode are also relaxed. The space of linear functionals $\hat{\mathcal{A}}$ on $\mathcal{A}$ is given the structure of a WHA by transposing all operations, the product, the coproduct, etc., with respect to a canonical pairing $\langle\, ,\, \rangle$ between $\mathcal{A}$ and $\hat{\mathcal{A}}$, defined with the help of the cells ${}^{(2)}F$. In particular the dual $3j$-symbols ${}^{(1)}\widetilde{F}$ appear in the coproduct in $\hat{\mathcal{A}}$. The dual algebra is a matrix algebra $\hat{\mathcal{A}} = \oplus_x\ \mathrm{Mat}_{\tilde{m}_x}$, with $\tilde{m}_x = \sum_{a,b} \tilde{n}_{ax}{}^b$. As we have seen, n_i and $\tilde{n}_x$ are (block) diagonalized in the same unitary basis, which ensures the equality of dimensions of the two (finite dimensional) algebras, $\sum_i (m_i)^2 = \sum_x (\tilde{m}_x)^2$. Identifying $\mathcal{A}$ and $\hat{\mathcal{A}}$, the cells ${}^{(2)}F$ are also seen as giving an invertible transformation relating two sets of double triangles, i.e., two bases in the algebra and one can define a scalar product consistent with the pairing $\langle\, ,\, \rangle$. Then the unitarity of the $3j$-symbols ${}^{(1)}F$, ${}^{(1)}\widetilde{F}$ implies that there are elements $\hat{e}_i$, $\hat{e}_j \in \mathcal{A}$ and $\hat{E}_x$, $\hat{E}_y \in \hat{\mathcal{A}}$, s.t. for a double triangle $e_k = e_{k;\alpha,\alpha}^{(ac),(ac)}$ or a dual double triangle $E_z = E_{z;\eta,\eta}^{(ac),(ac)}$, one has

$$\langle \hat{e}_i \otimes \hat{e}_j\, ,\, \triangle(e_k)\rangle \;=\; \sum_p N_{ij}{}^p\, \langle \hat{e}_p\, ,\, e_k\rangle = N_{ij}{}^k \tag{6.1}$$

$$\langle \hat{E}_x \otimes \hat{E}_y\, ,\, \hat{\triangle}(E_z)\rangle \;=\; \sum_w \widetilde{N}_{xy}{}^w\, \langle \hat{E}_w\, ,\, E_z\rangle = \widetilde{N}_{xy}{}^z\,. \tag{6.2}$$

In this way the two sets of integers N and $\widetilde{N}$ determine fusion rings in each algebra of the dual pair. In type I cases we can restrict the second equality to the subalgebras $\hat{N}$ and N^{ext}, selecting accordingly subalgebras of $\hat{\mathcal{A}}$. Note that (5.2) plays an analogous rôle for the *double* $\mathcal{D}(\mathcal{A})$ of the WHA $\mathcal{A}$.

These pairs of WHA can be given a field theoretic interpretation. The nimreps $n_{ia}{}^b$ provide the multiplicities of Cardy boundary fields and at the same time count the triangles, i.e., the states of the representation spaces V^i with fixed a, b. One can then combine the CVO $\phi_{ij}^k(z)$ with intertwiners $V^j \to V^k$ to define fields covariant with respect to the WHA algebra $\mathcal{A}$ [32]; for real arguments this gives a precise operator meaning to the boundary fields. Furthermore the physical (half-plane) bulk fields of the BCFT are defined as chiral compositions of the generalized CVO. The new constants in this construction, the *bulk-boundary* or *reflection* coefficients $R_a^{(j,\bar{j})}(p)$, are defined for $n_{pa}{}^a N_{j\bar{j}}{}^p \neq 0$. [‡] They are subject of equations [8], which in particular determine them explicitly for scalar fields in terms of F and ${}^{(1)}F$, see [32] for a general formula. All correlators are reduced to explicit linear combinations of conventional chiral blocks.

In precisely the same way one can define in the type I cases generalized CVO for the extended theory, which are covariant under a subalgebra of $\hat{\mathcal{A}}$, spanned by the dual double triangles with $x = a\,, a \in T$. These are "diagonal" covariant CVO, defined using the dual cells $\widetilde{F}$ or ${}^{(1)}\widetilde{F}$, restricted to labels in T, i.e., the $6j$-symbols $F^{\rm ext}$ introduced in the previous section.

The algebra $\mathcal{A}$ possesses an $\mathcal{R}$ matrix, i.e., has the structure of quasi-triangular WHA. Accordingly one introduces braiding matrices for the generalized CVO; they are diagonalized by the $3j$-symbols ${}^{(1)}F$, while the eigenvalues are phases determined by the scaling dimensions of the CVO. These braiding matrices were shown also to reproduce the Boltzmann weights [26, 32] of associated lattice models [27]. In general there is no natural way to turn $\hat{\mathcal{A}}$ into a quasi-triangular algebra even in the commutative $\widetilde{N}$ cases. Yet for type I cases there are matrices $B^{\rm ext}(\pm)$, diagonalized by $F^{\rm ext}$, which reproduce the braiding matrices of the CVO (or of the diagonal generalized CVO) of the extended theory.

Last but not least, the (commutative) duals of the $\hat{N}$ and $\widetilde{N}$ algebras, i.e., the *Pasquier algebra* [28] and its generalization, have been shown to be highly relevant in the determination of the OPE coefficients of the physical bulk fields. The structure constants $M_{(i,\alpha)(j,\beta)}{}^{(k,\gamma)} = \sum_a \psi_a^{(i,\alpha)}\, \psi_a^{(j,\beta)} \psi_a^{(k,\gamma)*}/\psi_a^1$ of the Pasquier algebra yield the OPE coefficients of scalar bulk fields normalised appropriately, [29, 30], while their analogues constructed out of $\Psi_x^{(j,\bar{j};\alpha,\alpha)}$ provide the same information for the modulus square of the OPE coefficients for all bulk fields of the theory, $\Phi_{(j,\bar{j};\alpha)}(z,\bar{z})$, $\alpha = 1, 2 \ldots Z_{j\bar{j}}$. The latter formula is truly nontrivial for the exceptional type II cases. These universal formulae arise from duality properties of bulk fields near boundaries [8, 33, 34, 2] [§] or defect lines [32].

In conclusion, despite its weakened axioms (as compared to the related quantum groups, say), this quantum algebra has many appealing features:

- its construction relies on combinatorial data (multiplicities) that encode the spectrum of the RCFT for various boundary conditions;
- it is consistent with the extended chiral algebra symmetry of type I cases;

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‡Taking different initial and final boundary labels in these compositions (and accordingly bulk-boundary coupling constants depending on pairs of such labels), leads to more general bulk fields, which produce for small $z-\bar{z}$ boundary-changing fields. It remains to check the consistency and relevance of this generalization.

§Rediscovered in the boundary CFT [33], as resulting from the sewing equations of [8], the algebra introduced in 1987 by Pasquier was called "classifying algebra" in [17], where its importance in this context was particularly stressed.

- its $6j$-, resp. $3j$- symbols F , $F^{\rm ext}$, and ${}^{(1)}F$ are intimately connected with the OPA of the RCFT;
- it incorporates in a natural way the truncation of representations inherent to RCFT;
- it enables one to construct generalized chiral vertex operators, giving precise operator meaning to the boundary and the physical (half-plane) bulk fields;
- its "weakness" is justified by the necessity of describing the boundary field degrees of freedom, e.g., the nontrivial "identity" space V^1 of dimension $|V^1| = tr(n_1)$ determines the set of vacuum states $|0\rangle \otimes |e^1_{aa}\rangle$ in the correlators of boundary and half-plane bulk fields;
- it determines the basic structure constants of the various operator algebras for any diagonal or non-diagonal RCFT.

This qualifies $\mathcal{A}$ as the natural quantum algebra of the RCFT!

7 Perspectives

There are various perspectives for further developments.

First the $\widehat{sl}(2)$ examples have to be completed. Although a lot of data are already available for these simplest non-trivial examples of the Ocneanu quantum symmetry, the elaboration of all the structures involved in the construction of the non-diagonal WHA $\mathcal{A}$ remains to be done. In general, the empirical observations or conjectures of sect. 5 and 6 have to be established on a firmer ground.

The discussion so far was mainly aimed at the description of the integrable WZW or the related minimal models. Other examples of finite dimensional WHA arise from the $c = 1$ Γ- orbifold CFT, where Γ is a subgroup of SU(2), and their generalizations to other groups [14]. The starting point is the complete set of boundary states, described by the representations of the quantum double $\mathcal{D}(\Gamma)$ of the finite group Γ. These are "Cardy type" solutions since there exists a Verlinde formula with a symmetric unitary S matrix; restricted to the subset of untwisted representations, it gives simply the representation ring of Γ (the graph algebra of the affine Dynkin diagrams in the SU(2) case and their generalizations). The cylinder partition functions can be represented either as finite sums in the rational characters with the Verlinde fusion multiplicities, or as infinite sums of Virasoro (or more generally W-) algebra characters, parametrised by the highest weights of the finite dimensional irreducible representations of SU(2) (or its generalization), see, e.g., [37] for the formulae restricted to the untwisted sector. These decompositions are analogous to (5.5) and (4.1), respectively. The coefficients in the second sum are "affine" analogs of the nimreps ${n_{ia}}^b$; in the untwisted sector $\{n_{i1}{}^a\}$ reproduce the multiplicities of the representations of Γ in the decomposition of the finite dimensional irreps $\{i\}$.

Another possible direction is the generalization of these ideas to non-rational CFT with a continuous spectrum.

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